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A Note on Modal Reverberation Times in Rectangular Rooms

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Summary

An equation for the reverberation time is derived for the normal modes in a rectangular room with different absorption coefficients assigned to the six surfaces. The method is based on the fact that the standing wave of an oblique mode can be split into eight plane sound waves travelling in welldefined directions. Thus, the concept of mean free path can be applied in connection with a normal room mode. Averaging the mean free path of a very large number of room modes leads to an asymptotic result that equals the mean free path known in statistical room acoustics for 2D and 3D diffuse sound fields.

1. Introduction

The solution to the wave equation in a rectangular room has been known since Lord Rayleigh's "The Theory of Sound" was published in 1877 [1, § 267], and it is found in almost every textbook on acoustics. Still, it is hard to find information in the literature about the reverberation time or the damping constant of the normal modes. An approximate solution is found in [2, eq. (9.5.24)], but that includes a dubious model that gives attenuation to a plane wave from a surface parallel with the direction of propagation. Actually, this author is not aware of any reference that presents a proper solution to the problem. The aim of this paper is to show how a solution can be derived by using the fact, that the standing wave in a rectangular room can be considered the result of up to eight plane waves propagating in well-defined directions.

2. Directions of sound propagation

A rectangular room with dimensions l_x, l_y, l_z is considered. Assuming rigid surfaces, i.e. reflections without phase delay, the solution to the wave equation can be written [3, eq. (III.18)],

$$\begin{aligned} p_n &= A \cos\frac{\pi x n_x}{l_x} \cos\frac{\pi y n_y}{l_y} \cos\frac{\pi z n_z}{l_z} \exp\left(\mathrm{j}\omega t\right) \qquad (1)\\ &= \frac{A}{8}\sum \exp\left[\mathrm{j}\pi\left(\pm\frac{n_x}{l_x}x \pm \frac{n_y}{l_y}y \pm \frac{n_z}{l_z}z\right)\right]\exp\left(\mathrm{j}\omega t\right). \end{aligned}$$

Here A is a constant, (n_x, n_y, n_z) are the modal numbers, and ω is the angular frequency. The summation is taken over all eight possible combinations of the + and − signs. The natural frequency of the mode is given by

$$f_n = \frac{c}{2}\sqrt{\left(\frac{n_x}{l_x}x\right)^2 + \left(\frac{n_y}{l_y}y\right)^2 + \left(\frac{n_z}{l_z}z\right)^2}, \qquad (2)$$

where c is the speed of sound. Equation (1) shows that the solution is a standing wave, but also that this can be interpreted as the interference between eight plane waves travelling in different directions. The wave number of the propagation along the x-axis is $k_x = \pi n_x / l_x$. The direction of propagation expressed as an angle relative to the x-axis is

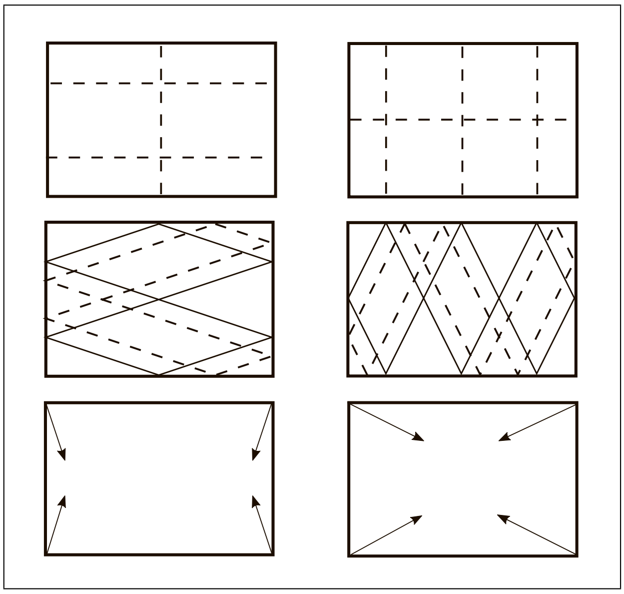

Figure 1. Two examples of tangential modes; (1, 2, 0) left and (3, 1, 0) right. Upper graphs: nodal lines, middle graphs: wave fronts in two positions with a short time delay. Lower graphs: directions of propagation, perpendicular to the wave fronts.

$$\begin{aligned} \cos\varphi_x &= \pm\frac{n_x/l_x}{\sqrt{\left(n_x/l_x\right)^2 + \left(n_y/l_y\right)^2 + \left(n_z/l_z\right)^2}}\\ &= \pm\frac{n_x c}{2 l_x f_n}, \qquad (3a) \end{aligned}$$

and analogue for the y-and z-directions,

$$\cos\varphi_y = \pm\frac{n_y c}{2 l_y f_n}, \qquad (3b)$$

$$\cos\varphi_z = \pm\frac{n_z c}{2 l_z f_n}. \qquad (3c)$$

If one modal number is zero, e.g. $n_z = 0$, the angle $\varphi_z = 90°$, and the sound propagation is perpendicular to the z-axis (a tangential mode). If two modal numbers are zero, e.g. $n_y = n_z = 0$, the sound propagation is parallel with the x-axis (an axial mode).

Two examples of tangential modes in a rectangular room are visualised in Figure 1. The nodal lines (actually vertical planes) represents the standing wave pattern, whereas the corresponding wave fronts represent the plane waves travelling in four different directions. The normal distance between parallel wave fronts is the wave length $\lambda = c/f$. The lower part of Figure 1 shows the four possible directions of propagation, which are perpendicular to the wave fronts. The direction of propagation and the directional cosines in equations (3) are displayed in Figure 2 in the case of an oblique mode.

3. Number of reflections and the mean free path

Being a rectangular room, it is convenient to consider sound reflections as a sound ray that continues as a straight line into the infinite surrounding of image rooms. The next step is to find how often each of the six surfaces of the room is active in the series of reflections, when one of the directions of propagation is followed. Any of the eight directions can be chosen, the result will

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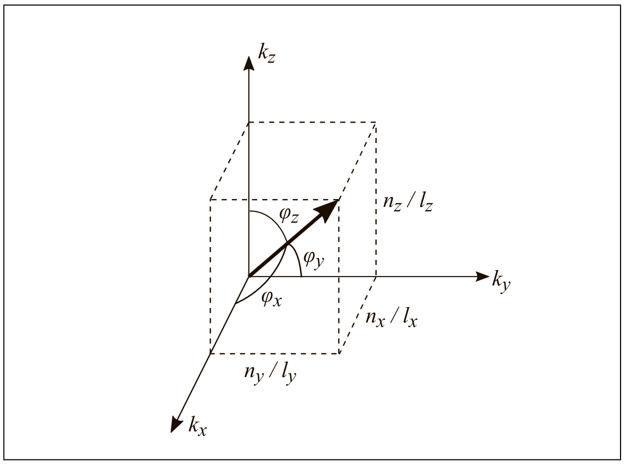

Figure 2. The direction of propagation and the directional cosines of an oblique mode.

be the same. If the total length of the ray is l_0, the number of reflections q_x in the x-direction is found by dividing the projection of l_0 on the x-axis by the room dimension l_x,

$$q_x = \frac{l_0}{l_x} \cos\varphi_x = \frac{l_0 n_x c}{2 l_x^2 f_n}, \tag{4}$$

and analogue for the y- and z-directions. The mean free path is then found as the total length l_0 divided by the total number of reflections,

$$l_m = \frac{l_0}{q_x + q_y + q_z} = \frac{2 f_n}{c}\left(\frac{n_x}{l_x^2} + \frac{n_y}{l_y^2} + \frac{n_z}{l_z^2}\right)^{-1}. \tag{5}$$

4. Modal reverberation time

The sound absorption coefficients of the six surfaces are denoted α_{x1}, α_{x2}, α_{y1}, α_{y2}, α_{z1} and α_{z2}. In fact, absorption coefficients that are not zero violate the presumption of rigid surfaces. Instead, angle independent absorption coefficients with zero phase angle are assumed. These assumptions are similar to those of geometrical acoustics and are the same as those required for specular angleindependent ray tracing, see [4]. As in geometrical acoustics, it is believed that they do not introduce serious problems into the result.

The reverberation time of the room modes can now be calculated in the same way as used in [5]. Following a representative wave in one of the directions introduced above, the two walls perpendicular to the x-axis are met q_x times, i.e. each wall provides a reflection $q_x/2$ times. A similar observation is made for the y- and z-directions.

The energy of the mode after all $N = q_x + q_y + q_z$ reflections along the length l_0 of the representative wave is

$$E_{N,n} = E_0 \left[\left(1-\alpha_{x1}\right)\left(1-\alpha_{x2}\right)\right]^{q_x/2} \left[\left(1-\alpha_{y1}\right)\left(1-\alpha_{y2}\right)\right]^{q_y/2} \cdot \left[\left(1-\alpha_{z1}\right)\left(1-\alpha_{z2}\right)\right]^{q_z/2}. \tag{6}$$

The length of propagation is $l_0 = ct$, where t is the time. By definition the reverberation time is the time, when the energy is reduced to $E_{N,n} = 10^{-6} E_0$. After insertion of (4) and the analogue expressions for q_y and q_z, the reverberation time T_n of a specific room mode (n_x, n_y, n_z) is found as

$$\begin{aligned} \ln\left(\frac{E_{N,n}}{E_0}\right) &= -6\ln(10) \\ &= \frac{c^2 T_n}{2 \cdot 2 \cdot f_n}\left[\frac{n_x}{l_x^2} \ln\left((1-\alpha_{x1})(1-\alpha_{x2})\right)\right. \\ &\quad + \frac{n_y}{l_y^2} \ln\left((1-\alpha_{y1})(1-\alpha_{y2})\right) \\ &\quad \left. + \frac{n_z}{l_z^2} \ln\left((1-\alpha_{z1})(1-\alpha_{z2})\right)\right], \end{aligned} \tag{7}$$

$$\begin{aligned} T_n = \frac{6\ln(10) \cdot 4 \cdot f_n}{-c^2} \cdot \left[\frac{n_x}{l_x^2} \ln\left((1-\alpha_{x1})(1-\alpha_{x2})\right)\right. \\ \left. + \frac{n_y}{l_y^2} \ln\left((1-\alpha_{y1})(1-\alpha_{y2})\right) + \frac{n_z}{l_z^2} \ln\left((1-\alpha_{z1})(1-\alpha_{z2})\right)\right]^{-1}. \end{aligned} \tag{8}$$

If we define the area of the surfaces as $S_x = l_y l_z$, $S_y = l_x l_z$, $S_z = l_x l_y$ and multiply the divident and divisor in (8) with the volume $V = l_x l_y l_z$, the modal reverberation time can be written as

$$\begin{aligned} T_n = \frac{55.3 \cdot v \cdot f_n}{-c^2} \cdot \left[\frac{n_x}{l_x} S_x \ln\left((1-\alpha_{x1})(1-\alpha_{x2})\right)\right. \\ + \frac{n_y}{l_y} S_y \ln\left((1-\alpha_{y1})(1-\alpha_{y2})\right) \\ \left. + \frac{n_z}{l_z} S_z \ln\left((1-\alpha_{z1})(1-\alpha_{z2})\right)\right]^{-1}. \end{aligned} \tag{9}$$

It is noted that the contribution of the absorption coefficient of a surface is not only proportional to the area of the respective surface, but also proportional to the index numbers of the mode and inversely proportional to the distance between the pair of parallel surfaces. This is a result, which is very different from the absorption of a surface in statistical room acoustics, and may be unique to the modal reverberation time.

This finding can be compared to the classical Eyring equation based on statistical room acoustics. Then we assume all absorption coefficients to be equal to the mean absorption coefficient α_m and from (8) we get the modal reverberation time,

$$\begin{aligned} T_n &= \frac{13.8 \cdot 4 \cdot f_n}{-c^2} \cdot \left[2\left(\frac{n_x}{l_x^2} + \frac{n_y}{l_y^2} + \frac{n_z}{l_z^2}\right) \ln\left(1-\alpha_m\right)\right]^{-1} \\ &= \frac{13.8 \cdot l_m}{-c \ln(1-\alpha_m)}, \end{aligned} \tag{10}$$

where l_m is the mean free path, as found above (5). In the statistical room acoustics the mean free path in a 3D diffuse field is known to be $l_m = 4V/S$, where S is the total surface area [6]. This is independent on room shape, but an equal distribution of all directions of propagation is an important assumption.

When the statistical mean free path is inserted in (10) we get Eyring's equation,

$$T_n = \frac{55.3 \cdot V}{-cS \ln(1-\alpha_m)}. \tag{11}$$

So, the modal reverberation time derived above is in agreement with the statistical Eyring equation if the modal mean free path equals $4V/S$ for a sufficient high density of room modes.

For comparison of the statistical mean free path to the modal mean free path, a few examples of edge ratios are chosen for rectangular rooms, ranging from a cubic room (1:1:1) to a very long room (8:1:1) or a very flat room (8:8:1), see Table I. The modal

Table I. The mean free path for 3D sound fields, calculated for some examples of room edge ratios.

	Edge ratio						
l_x	1	1.6	4	8	8	8	8
l_y	1	1.25	2	1	2	4	8
l_z	1	1	1	1	1	1	1
V	1	2	8	8	16	32	64
S	6	9.7	28	34	52	88	160
l_m (stat)	0.667	0.825	1.143	0.941	1.231	1.455	1.600
l_m (728 modes)	0.678	0.829	1.081	0.887	1.142	1.369	1.612
l_m ratio	1.017	1.005	0.946	0.942	0.928	0.941	1.007
l_m (511 modes)	0.680	0.822	1.051	0.881	1.108	1.300	1.503
l_m ratio	1.020	0.997	0.920	0.936	0.900	0.894	0.939

Table II. The mean free path for 2D sound fields, calculated for some examples of room edge ratios.

	Edge ratio						
l_x	1	1.25	1.6	2	3	4	8
l_y	1	1	1	1	1	1	1
S	1	1.25	1.6	2	3	4	8
U	4	4.5	5.2	6	8	10	18
l_m (stat)	0.785	0.873	0.967	1.047	1.178	1.257	1.396
l_m (80 modes)	0.807	0.895	0.986	1.064	1.201	1.310	1.708
l_m ratio	1.027	1.025	1.021	1.016	1.020	1.042	1.223

mean free path is calculated from (5) by averaging the result for all possible combinations of room modes from (1,0,0) to (8,8,8), i.e. the first 728 modes. It is emphasised that axial and tangential modes are included. If the averaging had been restricted to oblique modes only, the modal mean free path would be shorter. However, when all modes are included, the agreement with the statistical model is good for the examined edge ratios. The perfect agreement should give a ratio of unity. The maximum deviation between modal and statistical mean free path shown in Table I is about 7% (the l_m ratio 0.928), which is for the edge ratio (8:2:1). The reason for the deviations is the limited number of modes included in the averaging; increasing the number of modes would reduce the deviations. To prove this, the calculation has been repeated with all room modes up to (7,7,7), i.e. the first 511 modes, and as expected the agreement with the statistical mean free path is slightly worse when less room modes are used.

In statistical room acoustics we also have the mean free path in a diffuse 2D field; $l_m = \pi S/U$, where $S = l_x l_y$ is the surface area and $U = 2(l_x + l_y)$ is the perimeter [5]. Again we can do a comparison to the average modal mean free path by looking at a number of edge ratios and averaging the result for all possible combinations of room modes from (1,0) to (8,8), i.e. the first 80 modes. As seen in Table II the agreement is quite satisfactory, even with this very limited number of modes.

5. Discussion

A simplified model for the modal energy in a rectangular room was presented recently [5]. In that paper the author suggested a simplified method to estimate the energy and reverberation time of each single mode. The method, which was based on intuition, suggested that the energy after $N = 2(n_x + n_y + n_z)$ reflections is

$$E_{N,n} = E_0\left[\left(1-\alpha_{x1}\right)\left(1-\alpha_{x2}\right)\right]^{n_x}\left[\left(1-\alpha_{y1}\right)\left(1-\alpha_{y2}\right)\right]^{n_y} \cdot \left[\left(1-\alpha_{z1}\right)\left(1-\alpha_{z2}\right)\right]^{n_z}. \quad (12)$$

So, according to the simplified method the absorption of the surfaces is weighted by the modal numbers, only. This is in contrast to the model as derived above, in which the room dimensions also contribute to the weighting.

The reverberation time of a specific room mode (n_x, n_y, n_z) according to the simplified theory [5] is

$$T_n = \frac{13.8 \cdot 2\sqrt{(l_x n_x)^2 + (l_y n_y)^2 + (l_z n_z)^2}}{-c \ln\left([(1-\alpha_{x1})(1-\alpha_{x2})]^{n_x} \cdot [(1-\alpha_{y1})(1-\alpha_{y2})]^{n_y} \cdot [(1-\alpha_{z1})(1-\alpha_{z2})]^{n_z}\right)}. \quad (13)$$

In the special case of a cubic room ($l_x = l_y = l_z$) the equations (8) and (13) yield the same result; but in general the simplified theory will give too long reverberation times for the tangential and oblique modes. The reverberation time of the axial modes is always correct with both models.

As an example for comparison of the models we look at the same room as in [5] with dimensions $(l_x, l_y, l_z) = (4.32\,\text{m}, 3.38\,\text{m}, 2.70\,\text{m})$ and absorption coefficients $(\alpha_{x1}, \alpha_{x2}, \alpha_{y1}, \alpha_{y2}, \alpha_{z1}, \alpha_{z2}) = (0.05, 0.05, 0.10, 0.10, 0.15, 0.80)$. The reverberation times calculated with the simplified model and the new (exact) model are shown in Table III for the first 14 modes. There is no difference for axial modes, but some of the tangential and oblique modes have shorter reverberation time according to the exact model. However, the practical consequences are very limited, because in a 1/3 octave frequency band the decay will normally be dominated by the axial modes.

A comparison of the two models in terms of direction of propagation is given in Table IV. The angles are calculated for the first tangential modes in a room with edge ratio 1 : 1.5. This edge ratio is also used in Figure 1, and we see that the angle relative to the x-axis is 72° for the (1, 2, 0) mode and 27° for the (3, 1, 0) mode. In contrast, the simplified model suggested 53° and 13° , respectively.

Table III. The first 14 modes and their reverberation times calculated with the simplified (T_{ns}) and the exact (T_{ne}) model in the example room. The underlined values deviate from the exact values.

n_x	n_y	n_z	f_n [Hz]	T_{ns} [s]	T_{ne} [s]
1	0	0	39.7	3.4	3.4
0	1	0	50.8	1.3	1.3
0	0	1	63.6	0.1	0.1
1	1	0	64.5	1.4	1.3
1	0	1	75.0	0.2	0.1
2	0	0	79.5	3.4	3.4
0	1	1	81.4	0.2	0.1
1	1	1	90.6	0.2	0.2
2	1	0	94.3	1.8	1.5
0	2	0	101.6	1.3	1.3
2	0	1	101.8	0.4	0.2
1	2	0	109.1	1.2	1.2
2	1	1	113.7	0.4	0.2
3	0	0	119.2	3.4	3.4

Table IV. Examples of direction of propagation calculated with the simplified (φ_{xs}) and the exact (φ_{xe}) model. Tangential modes in room with edge ratio 1 : 1.5 (as in Figure 1). $l_x = 1.5 l_y$.

(n_x, n_y, n_z)	φ_{xs}	φ_{xe}
(3, 0, 0)	0°	0°
(3, 1, 0)	13°	27°
(2, 1, 0)	18°	37°
(1, 1, 0)	34°	56°
(1, 2, 0)	53°	72°
(1, 3, 0)	63°	77°
(0, 3, 0)	90°	90°

6. Conclusion

The normal modes in a rectangular room can be associated with the interference of plane, propagating sound waves, and thus the concept of mean free path can be applied. This allows the calculation of the attenuation of a room mode as a function of the absorption coefficients of the surfaces. An equation for the reverberation time has been derived for the normal modes in a rectangular room with different absorption coefficients assigned to the six surfaces.

When the absorption coefficient is the same on all surfaces, the modal reverberation time can be compared to the Eyring equation, which is known from statistical room acoustics. It is shown that the averaging over a large number of modes yields asymptotically the same mean free path as known from statistical room acoustics for 2D and 3D diffuse sound fields. However, each room mode is associated with an individual mean free path, and thus the modal reverberation time varies strongly from one mode to the next.

The results have also been compared to a simplified model for the modal reverberation time. In general the simplified model will give too long reverberation times for the tangential and oblique modes. The simplified model gives correct results for the axial modes in all cases and for the other modes when the edge ratios are close to (1:1:1).

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